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Linear Spaces, Subspaces and Hamel Bases

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Abstract: A linear space is a collection of objects called vectors which may be added together and multiplied by numbers, called scalars. Scalars are taken to be real numbers, sometimes complex numbers, rational numbers or generally any field. Linear spaces are the subject of linear algebra and are well characterized by their dimension, which roughly speaking specifies the number of independent directions in the space.

1. Introduction

The focus of this paper is on linear space, linear hull, Hamel bases, dimensionality of linear space, some theorms on dimensionality of linear space.

1.1 Definition : linear space [1] : A linear space over C (complex linear space) is non empty set X with a funtion + on $X \times X$ into X, and a function . on $C \times X$ into X such that for all complex λ, μ and elements (vectors) x, y, z in X we have (1) $x + y = y + x$ (2) $x + (y + z) = (x + y) + z$ (3) there exists $\theta \in X$ such that $x + \theta = x$ (4) there exists $-x \in x$ such that $x + (-x) = \theta$ (5) $1.X = x$ (6) $\lambda(x + y) = \lambda x + \lambda y$ (7) $(\lambda + \mu)x = \lambda x + \mu x$ (8) $\lambda(\mu x) = (\lambda\mu)x$. An equivalent way of defining a linear space is that it is an additive abelian group w.r.t addition i.e (1) to (4) holds, for which also scalar multiplication is defined such that (5) to (8) holds. The element θ is called zero, neutral element or origin in X. It is easy to see that θ and $-x$ are unique.

Example : (1) C is complex linear space with usual addition and multiplication for complex numbers.

(2) $\mathbb{R}^n$ becomes a real linear space if we define coordinatewise operations as : $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$ where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and λ is real.

(3) Let s be the space of all the sequence (x_n) , then s becomes linear space under definitions $(x_n) + (y_n) = (x_n + y_n)$, $\lambda(x_n) = (\lambda x_n)$

1.2 Linear map and isomorphism[2] : let X ,Y be linear spaces over the scalar field . A map $f : X \to Y$ is called linear if $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for all scalars λ, μ and all $x, y \in X$. An isomorphism $f : X \to Y$ is bijective linear map then we say X and Y are isomorphic if there is an isomorphism $f : X \to Y$

Example: prove that $f : \mathbb{R}^3 \to \mathbb{R}^3$, given by $f(x) = (x_2, -x_1, x_2)$ is an isomorphism

Proof: First we will prove that f is a linear map . So consider scalars λ , μ and x , $y \in \mathbb{R}^3$ i.e $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$$
\begin{aligned}
\Rightarrow f(\lambda x + \mu y) &= f(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \lambda x_3 + \mu y_3) \\
&= (\lambda x_2 + \mu y_2, -(\lambda x_1 + \mu y_1), \lambda x_3 + \mu y_3) \\
&= (\lambda x_2 - \lambda x_1, \lambda x_3) + (\mu y_2, -\mu y_1, \mu y_3) = \lambda(x_2, -x_1, x_3) \\
&= \lambda f(x) + \mu f(y)
\end{aligned}
$$

Next, one- one , let $f(x) = f(y) \Rightarrow f(x_1, x_2, x_3) = f(y_1, y_2, y_3)$

$$
\begin{aligned}
&\Rightarrow (x_2, -x_1, x_3) = (y_2, -y_1, y_3) \\
&\Rightarrow x_2 = y_2, x_1 = y_1, x_3 = y_3
\end{aligned}
$$

We find $\Rightarrow x = y$

Onto: Let $y \in \mathbb{R}^3$ be such that we find $x \in \mathbb{R}^3$ so that $f(x) = y \Rightarrow (x_2, -x_1, x_3) = (y_1, y_2, y_3)$

$$\Rightarrow x_2 = y_1, x_1 = -y_2, x_3 = y_3$$

Thus $f(-y_2, y_1, y_3) = (y_1, y_2, y_3)$

$\Rightarrow$f is onto. Hence f is isomorphism.

2. Subspaces ,Dimensionality

2.1 Definition: A subspace M in a linear space X is a non empty subset of X such that $\lambda x + \mu y \ \varepsilon$ M whenever x, y ε M, for all $\lambda,\mu \ \varepsilon$ C. We see that if $\{M_\alpha\}$ is a family of subspaces then $\cap M_\alpha$ is also a subspace.

2.2 linear hull [3]: let S be a subset of linear space X. Then l. Hull(S), linear hull of S, is intersection of all subspaces containing S. Some terms such as 'span of S' or 'subspaces generated by S' are also used for linear Hull of S

2.3 Linear independence: A finite subset $(x_1, x_2, \ldots, x_n)$ of X is called linearly independent set iff a relation of the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \theta$ implies $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. If a finite subset is not linearly independent then it will be called linearly dependent . An arbitrary subset of X is called linearly independent iff every one of its finite subsets is linearly independent.

2.4 Hamel Base: A subset B of X is called Hamel base for X iff B is linearly independent set and l .hull(B) = X

2.5 Dimensionality [4]: A linear space X is called finite dimensional iff X has a finite Hamel base i.e B is a finite dimensional set which is Hamel base and the number of elements in Hamel Base is called dimension of X.

If X is not finite dimensional then it is called infinite dimensional.

2.6 Theorem: Linear space C^n has dimension n.

Proof: Consider the linear space C^n and let $e_i = (0,0,\ldots,1,0,\ldots)$,where 1 is in i-th place and there are zeros in other n-1 places. The set $(e_1, e_2,\ldots, e_n)$ is called set of unit vectors in C^n.

Consider $\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n = \theta$

$\Rightarrow \lambda_1 (1,0,\ldots0) + \lambda_2(0,1,\ldots0) + \ldots + \lambda_n(0,0,\ldots,1) = =0$

$\Rightarrow (\lambda_1, \lambda_2,\ldots, \lambda_n) = (0,0,\ldots,0) \Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

$\Rightarrow$Set B = $(e_1, e_2,\ldots ,e_n)$ is linearly independent.

Now, clearly l.hull(B) $\subset C^n$, let us take $x = (x_1, x_2,\ldots, x_n) = x_1 (1,0,0,\ldots,0) + x_2(0,1,0,\ldots,0) + \ldots + x_n(0,0,\ldots,1) = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$

$\Rightarrow x \ \varepsilon$ l.Hull(B)

$\Rightarrow$l.Hull(B) = $C^n \Rightarrow$B is Hamel Base for C^n .since B has n elements.

$\Rightarrow C^n$ has dimension n.

2.7 Theorem [5]: If X have a Hamel Base with n elements. Then any set of n + 1 elements in X is linearly dependent.

Proof: If n = 1 and {b} is Hamel Base then for each x_1, x_2in X we have $x_1 = \lambda_1 b$, $x_2 = \lambda_2 b$

If $\lambda_1 \lambda_2 = 0 \Rightarrow \lambda_1 = 0$ or $\lambda_2 = 0 \Rightarrow$either $x_1 = \theta$ or $x_2 = \theta \Rightarrow \{x_1, x_2\}$ is linearly dependent. If $\lambda_1 \lambda_2 \neq 0 \Rightarrow \lambda_1 \neq 0, \lambda_2 \neq 0$ and $\lambda_2 x_1 - \lambda_1 x_2 = \lambda_2(\lambda_1 b) - \lambda_1 (\lambda_2 b) = 0$ where $\lambda_1, \lambda_2 \neq 0$

$\Rightarrow \{x_1, x_2\}$ is linearly dependent. Thus result is true for n= 1. Consider the case n = 2, finishing the proof by induction. Take n = 2, B = $\{b_1, b_2\}$ a Hamel Base .

Let S = (x_1, x_2, x_3) be any 3- element set in X then $x_i = \lambda_{i1} b_1 + \lambda_{i2} b_2$ (I = 1,2,3)

Consider the subspace M = l.Hull(b_1). If all of $x_1, x_2, x_3 \ \varepsilon$ M then, since $\{b_1\}$ is Hamel base for M, the case n = 1 shows that element set $\{x_2, x_3\}$ is linearly Dependent. If however x_1, x_2, x_3 are not all in M $\Rightarrow x_3 \notin M$ implies $\lambda_{32} \neq 0$, for otherwise $x_3 = \lambda_{31} b_1 \ \varepsilon$ M, contrary to hypothesis. For i = 1,2, define $y_i = x_i - \lambda_{i2} x_3/ \lambda_{32} = \lambda_{i1} b_1 + \lambda_{i2} b_2 - \lambda_{i2}(\lambda_{31} b_1 + \lambda_{32} b_2)/ \lambda_{32} \ \varepsilon$ M from case n = 1, two element set $\{y_1, y_2\}$ is linearly dependent i.e there exist u_1, u_2 not both zero such that $u_1 x_1 + u_2 x_2 + \lambda x_{3 = \theta}$ where λ depends on $u_1, u_2, \lambda_{12}, \lambda_{22}, \lambda_{32}$ .Hence we see S is linearly dependent which proves theorem for n = 2. Thus by using idea of the case n = 2, it is easy to finish the proof inductively.

2.8 Theorem: Let X be finite dimensional. Then all the Hamel Bases for X have the same number of elements.

Proof: Let B is a Hamel Base with n elements and let B' be another Hamel Base for X. B' must be finite , otherwise it could have n + 1 linearly independent elements, contrary to theorem 2.7. If B' has m elements, then we have to prove that m = n . For if m > n or m < n, we contradict theorem 2.7, since B,B' are both bases.

2.9 Theorem: If X is finite dimensional with dimension n, then X is isomorphic to C^n.

Proof: Since X is finite dimensional with dimension n there is Hamel Base (say) $\{b_1, b_2, \ldots, b_n\}$. If $x \varepsilon X \Rightarrow x = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n$ for some scalars λ_i .The λ_i are unique ,for if $x = u_1 b_1 + u_2 b_2 + \ldots + u_n b_n$

$$\Rightarrow \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n = u_1 b_1 + u_2 b_2 + \ldots + u_n b_n$$

$$\Rightarrow (\lambda_1 - u_1) b_1 + (\lambda_2 - u_2) b_2 + \ldots + (\lambda_n - u_n) b_n = \theta \Rightarrow \lambda_i = u_i \ (i \le i \le n)$$

By linear independence of b_i. Now, define a map $f : X \to C^n$ as: let $x \varepsilon X$

$\Rightarrow$ $x = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_n b_n$ for unique scalars λ_i

Now, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \varepsilon C^n$

$\Rightarrow$define $f(x) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is well defined map. Clear f is bijective and it is easy to check $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for scalars α,β and x,y ε X . Hence f is an isomorphism .

3. Convex, balanced, absolutely convex, absorbent

3.1 Definition [6]: let E be a non-empty subset of linear space X.

1. E is called convex iff x,y ε E and $\lambda + \mu = 1$, with $\lambda \ge 0$, $\mu \ge 0$, imply $\lambda x + \mu y \ \varepsilon$ E
2. E is called balanced iff x ε E and $|\lambda| \le 1$ imply $\lambda x \ \varepsilon$ E
3. E is called absolutely convex iff x,y ε E and $|\lambda| + |\mu| \le 1$ imply $\lambda x + \mu y \ \varepsilon$ E
4. E is called absorbent iff to every x ε X there corresponds a number p = p(x) > 0 such that if $|\lambda| \le p$ then $\lambda x \ \varepsilon$ E

3.2 Theorem [7]: Denote by d the metric on C^n given by $d(x,y) = \left(\sum_{k=1}^{n} |x_k - y_k|^2\right)^{1/2}$ for each $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in C^n

Then any closed sphere $S[a,r] = (x \ \varepsilon \ C^n / d(x,a) \le r)$ of centre a ε C^n and radius r > 0, is a convex subset of C^n

Proof: Take x,y ε S[a,r], $\lambda + \mu = 1$, $\lambda \ge 0$, $\mu \ge 0$ then $d(x,a) \le r$, $d(y,a) \le r$

Now, $d(\lambda x + \mu y, a) = \left(\sum_{k=1}^{n} |\lambda x_k + \mu y_k - a_k|^2\right)^{1/2}$

$= \left(\sum_{k=1}^{n} |\lambda x_k + \mu y_k - (\lambda + \mu) a_k|^2\right)^{1/2} = \left(\sum_{k=1}^{n} |\lambda(x_k - a_k) + \mu(y_k - a_k|^2\right)^{1/2}$

$\le \left(\sum_{k=1}^{n} |\lambda(x_k - a_k|^2\right)^{1/2} + \left(\sum_{k=1}^{n} |\mu(y_k - a_k|^2\right)^{1/2} = \lambda\, d(x,a) + \mu\, d(y,a) \le \lambda r + \mu r = r$

On using Minkowski's inequality. Hence we have show that $d(\lambda x + \mu y, a) \le r$ which implies $\lambda x + \mu y \ \varepsilon$ S[a,r] so S[a,r] is convex.

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